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THE SERIES OF POLYNOMIALS IN THE PROBLEM OF THREE BODIES

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SUMMARY

The series of polynomials in the problem of three bodies converging for any real moment of time are investigated by numerical methods. The convergence coefficients, which transform the Taylor series with some finite convergence circle in a series of polynomials converging for any point within the Mittag-Leffler rectilinear star of generative function are given in Section 1.

Such power-polynomial series by mean anomaly are constructed in Section 2 for the elliptic three-body problem.

For the sake of comparison, the Sundman series, related to the same problem, are analyzed in Section 3.

Finally, the series of power-polynomials by the variable regularizing the double collisions is dealt with in Section 4.

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INTRODUCTION.

The methods discussed in the present work are mainly half a century old, and possibly more. However, the utilization of fast computers, only made possible in our time, allows a different approach to these methods, and in numerous cases a significant broadening of the area of their application.

Assume that $f(\omega)$ is a certain analytical function, locally given by the Taylor series

$$f(\omega) = a_0 + a_1\omega + a_2\omega^2 + \dots, \tag{1}$$

converging in a certain circle with center at coordinate origin. According to the most general theory on analytic continuation, brought forth by Littag-Leffler in 1898 [12, 13], $f(\omega)$ may be represented in any region, internal relative to the rectilinear star of that function, by uniformly converging series of polynomials whose coefficients linearly degenerate through the coefficients of series (1). In other words, there exists for any point ω inside the Mittag-Leffler rectilinear star a sequence of polynomials $f_n(\omega)$ $(n = 1, 2, \ldots)$, converging to $f(w)$, where

$$f_n(\omega) = c_0^{(n)} a_0 + c_1^{(n)} a_1 \omega + \ldots + c_{m_n}^{(n)} a_{m_n} \omega^{m_n}. \tag{2}$$

The convergence factors $c_k^{(n)}$ are not dependent upon the form of function $f(\omega)$ and they are in their turn the coefficients of polynomials

$$g_n(\omega) = c_0^{(n)} + c_1^{(n)} \omega + \ldots + c_{m_n}^{(n)} \omega^{m_n}, \tag{3}$$

uniformly converging to the value of the function

$$g(\omega) = \frac{1}{1-\omega}, \tag{4}$$

provided only ω does not assume real values from 1 to ∞. Volterra [30] called at once attention to the significance of Mittag-Leffler theorem for the problem of dynamics, noting that if the coordinates of the considered dynamic system are analytical functions of time or of a certain equivalent variable in the region encompassing the entire real axis, they may be expanded in series of polynomials converging for any real value of that variable.

The more rapid the convergence of polynomials' (3) sequence to function $g(\omega)$, the faster, generally speaking, the convergence of polynomials (2) to function $f(\omega)$. Indeed, it follows from the Cauchy integral

$$f(x) - f_n(x) = \frac{1}{2\pi i} \int_\Gamma \left[\frac{1}{1-\frac{x}{\omega}} - g_n\left(\frac{x}{\omega}\right) \right] \frac{f(\omega)}{\omega} d\omega, \tag{5}$$

where $\underline{x}$ is any point of function's $f(\omega)$ rectilinear star, and Γ is the closed curve of length L, located within the star, surrounding the point $\underline{x}$ and such that any ray, emanating from the origin, intersects this curve at one and only point. When ω describes the curve Γ, x/ω describes the path not having any common point with the part of the real axis from 1 to ∞. This is why there is for any $\varepsilon > 0$ $N = N(\varepsilon)$ such that for $n > N$ the absolute value of the difference standing in the integrand (5), will be $<\varepsilon$. Hence precisely follows

$$|f(x) - f_n(x)| \leqslant \frac{\varepsilon L}{2\pi} \max \left|\frac{f(\omega)}{\omega}\right|_\Gamma. \tag{6}$$

Picard [18] and Painleve [16] pointed in 1899 to the close link between the expansion of analytical functions in series of polynomials and the integration of differential equations by the Cauchy-Lipschitz method. In this method, expounded in detail by Picard [20], for example, the following sequence of functions $x_i^{(n,n)}$ is constructed

$$\left.\begin{aligned}
x_i^{(n,1)} &= x_i^{(0)} + \varphi_i(0;\ x_k^{(0)}) \frac{\omega}{n}, \\
x_i^{(n,2)} &= x_i^{(n,1)} + \varphi_i\left(\frac{\omega}{n};\ x_k^{(n,1)}\right) \frac{\omega}{n}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
x_i^{(n,n)} &= x_i^{(n,n-1)} + \varphi_i\left(\frac{n-1}{n}\omega;\ x_k^{(n,n-1)}\right) \frac{\omega}{n},
\end{aligned}\right\} \tag{7}$$

which converges uniformly to the solution $x_i(\omega)$ of the system of differential equations (as $n \to \infty$)

$$\frac{dx_i}{d\omega} = \varphi_i(\omega;\ x_1, \ldots, x_m) \quad (i = 1, 2, \ldots, m) \tag{8}$$

with initial conditions $x_i(0) = x_i^{(0)}$. If the right-hand parts of these equations do not clearly depend on ω and are polynomials from variables x_i, the Cauchy-Lipschitz method does not provide the possibility of determining these regions.* But if, in particular, it is known from either additional considerations that functions $x(\omega)$ are bounded in absolute value from above over the entire real axis, the series of polynomials constructed for them by the Cauchy-Lipschitz method converge for any real value of ω.

Sundman [27] demonstrated in 1912 that if in the three-body problem the vector of areas is not zero, the rectangular coordinates of the bodies, the velocity components and the time are analytical functions in an infinite band of width 2Ω and symmetrical relative to the real axis of the complex plane of the variable ω, regularizing the double collisions. Further Sundman applied the Poincaré's mapping of the indicated band on a circle of unitary radius of the plane

$$\omega = \frac{2\Omega}{\pi} \ln \frac{1+\theta}{1-\theta}, \qquad \theta = \frac{\exp(\pi\omega/2\Omega) - 1}{\exp(\pi\omega/2\Omega) + 1} \tag{9}$$

and obtained the representation of the general solution of the three-body problem in the form of power series by θ, converging for any $|\theta| < 1$, and by the same token, also for any real moment of time t. It was clear, however, that series of such a type are of little validity for the clarification of the entire pattern of motion in the problems of dynamics. But it remained unknown, whether or not they could be utilized for the numerical solution of the three-body problem. In 1933 Belorizky [4] gave a negative response to this question. He revealed on the example of partial Lagrange solutions of the three-body problem an extremely slow convergence of Sundman series (in the cases considered by him it is necessary to take from $10^{8 \cdot 10^4}$ to $10^{8 \cdot 10^6}$ terms of series to obtain one correct sign over less than one sixth of the convolution). In 1953-1955 Vernić [28, 29] undertook the attempt to revise the Belorizky results and practically utilize the general solution of the three-body problem. However, he admitted a whole series of imprecisions and, in particular, erroneously asserted that the series by powers ω converge on the entire plane ω. These errors, discovered by G. A. Merman in 1956 [2], depreciated nearly entirely the Vernić works.

In his work Sundman himself did not mention the possibility of representing the general solution of the three-body problem in the form of polynomial series. To that possibility, stemming directly from his fundamental theorem, pointed directly Sundman's contemporaries, and first of all Picard [19]. In later literature only separate reminders are encountered (for example, Happel, 1941, [8]). In our times this question was again raised in the Merman 1958 work [3], where the equations for the three-body problem are reduced to the polynomial form and for the case of division of three-body motion in two nearly Keplerian motions an estimate of error, resulting from the substitution of the exact solution by Cauchy-Lipschitz polynomials, is constructed.

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* for it leads directly to the expansion of functions $x_i(\omega)$ in series of polynomials converging in respective rectilinear Mittag-Leffler stars.

These investigations served as stimulus to conducting the present work. Its object is the effective construction of series of polynomials in the three-body problem and the study of the possibilities of their utilization for the numerical solution. Preliminary analysis has shown that the algorithm of polynomial sequence construction by the Cauchy-Lipschitz method is considerably more complex than that for type-(2) polynomials. Moreover, the former converge very slowly. It is true, however, that the improvement of convergence may be attained by the application of the Cauchy-Lipschitz method's variant proposed by Picone in 1932 [21]. In the assumption that the right-hand parts of Eqs.(8) have continuous partial derivatives by their arguments to the $(\nu + 1)$-st order inclusive $(\nu > 1)$, Picone applies the Cauchy-Lipschitz method to the system

$$\frac{d^j x_i}{d\omega^j} = \varphi_i^{(j)}(\omega;\ x_1, \ldots, x_m) \quad (i=1, 2, \ldots, m;\ j=1, 2, \ldots, \nu), \tag{10}$$

where

$$\varphi_i^{(1)} = \varphi_i, \quad \varphi_i^{(j+1)} = \frac{\partial \varphi_i^{(j)}}{\partial \omega} + \sum_{k=1}^{m} \frac{\partial \varphi_i^{(j)}}{\partial x_k} \varphi_k,$$

and constructs the sequence of functions $x_i^{(n,n)}$ in the following form:

$$\left.\begin{aligned}
x_i^{(n,1)} &= x_i^{(0)} + \sum_{j=1}^{\nu} \frac{1}{j!} \varphi_i^{(j)}(0;\ x^{(0)}) \left(\frac{\omega}{n}\right)^j, \\
x_i^{(n,2)} &= x_i^{(n,1)} + \sum_{j=1}^{\nu} \frac{1}{j!} \varphi_i^{(j)}\left(\frac{\omega}{n};\ x_k^{(n,1)}\right) \left(\frac{\omega}{n}\right)^j, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_i^{(n,n)} &= x_i^{(n,n-1)} + \sum_{j=1}^{\nu} \frac{1}{j!} \varphi_i^{(j)}\left(\frac{n-1}{n}\omega;\ x_k^{(n,n-1)}\right) \left(\frac{\omega}{n}\right)^j.
\end{aligned}\right\}$$

Picone estimates the error stemming from the substitution of the exact solution $x_i(\omega)$ of the function $x_i^{(n,n)}$, and finds that this error decreases with the rise of <u>n</u> as $1/n^\nu$, while in the standard Cauchy-Lipschitz method, the rate of error decrease is proportional to $1/n$. However, the Picone variant is practically effective only in the case of simple right-hand parts of Eqs.(8), which cannot be said of the right-hand parts of the equations of motion of the three-body problem, reduced to the polynomial form. All these causes compel us to renounce the Cauchy-Lipschitz polynomials and turn to type-(2) polynomials. The coefficients a_k of the Taylor series are found relatively simply from recurrent relations, while those of convergence $c_k^{(n)}$ could be computed only once, recorded on a magnetic tape and then be object of utilization at any time.

All the computations connected with the present work were made with the aid of computer M-20. By its contents the work itself is divided in four sections. In the first section the convergence factors $c_k^{(n)}$ are found, in the second the series of polynomials are constructed for the two-body problem, in the third section the Sundman series for the two-body problem are investigated and in the fourth the series of polynomials in the three-body problem are computed.

The setup of Section 3 is called for by the following circumstance. As was shown by Belorizky, the exclusively slow convergence of Sundman series is explained by two causes. The first of them consists in th character itself of the Poincaré transformation, setting in correspondence to even small positive values of ω the values of θ quite close to the unity. (Note that then the interval (Ω, ∞) is transferred into the interval$(\approx 0.0656,1)$ of the axis θ. The second cause is the smallness of the number Ω.* This estimate is valid for any types of motion in the three-body problem and it is unquestionably strongly underrated. In this connection it appeared to be of interest to corroborate the qualitative estimates of Belorizky and to construct the Sundman series for the two-body problem , where the quantity Ω, the minimum distance of singular points in the plane ω from the real axis, is also precisely known. Obviously, the requirement of transition to the regularizing variable drops off, and it is sufficient to assume for the value of ω, for example, the mean anomaly M.

Section 1. Computation of Convergence Factors

Contrary to the coefficients of Taylor series, those of polynomial series and their powers have an innumerable multiplicity of values. A large quantity of various $c_k^{(n)}$ is known in literature (for example, Mittag-Leffler, 1900-1920 [14]; Le Roy, 1900 [10], Lindelöf, 1903 [11], Perron, 1922 [17], and their number could be increased without difficulty. Some of these $c_k^{(n)}$ were computed by us. However, the sequence of polynomials (3), constructed with their aid, converges too slowly, and these $c_k(n)$ were rejected. There is hardly any sense in bringing them up here. Generally, ideal would be such values of $c_k^{(n)}$, which assure the maximum rapid convergence to function $g(\omega)$ for the minimum order of polynomials (3).

One of the most practical expansions of function $g(\omega)$, encountered in literature, is the Goursat expansion (see Goursat,1903 [7]), constituting the result of application of the Cauchy-Lipschitz method to the differential equation

$$\frac{dg}{d\omega} = g^2, \tag{13}$$

which determined function (4) at the initial condition $g(0) = 1$. It is natural to attempt to generalize the Goursat method and to apply to Eq.(13) the Cauchy-Lipschitz method with the Picon variant (modification).

Assume that ν is an arbitrary, but fixed natural number $(\nu \geqslant 1)$. The system (10) will be in the given case

$$\frac{d^j g}{d\omega^j} = j!\, g^{j+1} \quad (j = 1, 2, \ldots, \nu). \tag{14}$$

With the aid of recurrent expressions

$$G_0(\omega) = 1, \qquad G_{n+1}(\omega) = \sum_{k=0}^{\nu} \omega^k [G_n(\omega)]^{k+1} \tag{15}$$

let us introduce the polynomials

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* Note that in his estimate Beloritzky was resting on the estimate of Ω made by Sundman.

$$G_n(\omega) = \sum_{k=0}^{m_n} b_k^{(n)} \omega^k, \tag{16}$$

of which the coefficients are whole positive numbers, and the power m_n is determined by the formula

$$m_n = (\nu + 1)^n - 1. \tag{17}$$

It is not difficult to be convinced that polynomials (3), sought for, are linked with polynomials (16) by the relation

$$g_n(\omega) = G_n\left(\frac{\omega}{n}\right) \tag{18}$$

and consequently,

$$c_k^{(n)} = \frac{1}{n^k} b_k^{(n)}. \tag{19}$$

Further, it is obviously seen that if a certain function $\phi(\omega)$ satisfies the functional equation

$$\sum_{k=0}^{\nu} \omega^k \cdot \varphi\left(\sum_{k=0}^{\nu} \omega^{k+1}\right) = \sum_{k=0}^{\nu} \omega^k [\varphi(\omega)]^{k+1}, \tag{20}$$

it must also be satisfied by the function

$$\psi(\omega) = \sum_{k=0}^{\nu} \omega^k [\varphi(\omega)]^{k+1}. \tag{21}$$

But, polynomial $G_1(\omega)$ satisfies Eq.(20). Consequently, all the polynomials $G_n(\omega)$ satisfy it also, i. e.,

$$\sum_{k=0}^{\nu} \omega^k \cdot G_n\left(\sum_{k=0}^{\nu} \omega^{k+1}\right) = \sum_{k=0}^{\nu} \omega^k [G_n(\omega)]^{k+1}. \tag{22}$$

Combining (15) and (22), we obtain

$$G_{n+1}(\omega) = \sum_{k=0}^{\nu} \omega^k \cdot G_n\left(\sum_{k=0}^{\nu} \omega^{k+1}\right). \tag{23}$$

Therefore, for the determination of coefficients $b_k^{(n)}$ we may utilize any of the three relations (15), (22) and (23). In particular, relation (22) is interesting in that it contains the coefficients of only one polynomial and it thus does not require the preservation in computer's memory of coefficients of the preceding polynomial. Per contra, relation (23) leads to a linear connection between the coefficients of two neighboring polynomials, and on the strength of that, they were given preference.

If we equate the coefficients at identical powers ω when substituting directly (16) into (23), we obtain

$$b_m^{(n+1)} = \sum_{k=k_1}^{k_2} d_{m-k}^{(k+1)} b_k^{(n)} \qquad (m=1,\ 2,\dots,\ m_{n+1}), \tag{24}$$

when the limits of summation are defined by formulas

$$k_1 = \max\left\{0,\ m - \left[\frac{\nu(m+1)}{\nu+1}\right]\right\}, \qquad k_2 = \min\{m,\ m_n\}, \tag{25}$$

and $d_k^{(m)}$ are positive whole numbers serving as coefficients in the expression

$$\left(\sum_{k=0}^{\nu} \omega^k\right)^m = \sum_{k=0}^{\nu m} d_k^{(m)} \omega^k. \tag{26}$$

When operating with computers, it is more practical to handle outright the coefficients $c_k^{(n)}$. For them the law (24) will be written in the form

$$c_m^{(n+1)} = \sum_{k=k_1}^{k_2} \left(\frac{n}{n+1}\right)^k \frac{d_{m-k}^{(k+1)}}{(n+1)^{m-k}} c_k^{(n)} \qquad (m=1,\ 2,\dots,\ m_{n+1}). \tag{27}$$

As to the numbers $d_k^{(m)}$, it follows from (26) that

$$d_k^{(m+1)} = \sum_{l=\max\{0,\ k-\nu\}}^{\min\{k,\ \nu m\}} d_l^{(m)} \qquad (k=1,\ 2,\dots,\ \nu m+\nu) \tag{28}$$

and the solution of this difference equation will be

$$d_k^{(m)} = \sum_{\lambda=0}^{\left[\frac{k}{\nu+1}\right]} (-1)^\lambda C_m^\lambda C_{k+m-1-\lambda(\nu+1)}^{m-1} \qquad (k=0,\ 1,\dots,\ \nu m). \tag{29}$$

When computing $d_k^{(m)}$ the equality

$$d_k^{(m)} = d_{\nu m-k}^{(m)} \quad \left(k=0,\ 1,\dots,\ \left[\frac{\nu m}{2}\right]\right). \tag{30}$$

was also utilized.

Formulas (27) and (29) fully resolve the problem of finding the convergence multipliers chosen by us, $c_k^{(n)}$. Note that all $\nu + 1$ coefficients of polynomials $g(\omega)$ and the first $\nu + 1$ youngest coefficients of polymomials $g_1(\omega)$ are equal to the unity ($c_k^{(n)} = 1$ for $k = 0, 1, \dots, \nu$ and $n > 1$). Subsequent coefficients decrease monotonically through the value $c_{m_n}^{(n)} = n^{-m_n}$. The initial rate of this decrease diminishes as the number n of the polynomial increases for a fixed ν, and with the increase of the number $\bar{\nu}$ for a fixed n. There is obviously no necessity to compute all the $c_k^{(n)}$ to $k = m$, and we may stop at the number k giving a negligibly small term in the polynomials (2). The introduction of scale factors into the linear law (27) allows us to materialize the computation of $c_k^{(n)}$ for any numbers k. But, for the sake of simplicity, we limited ourselves to the computation of $c_k^{(n)}$ through the fulfillment of one of the conditions: $k = m$ or $c_k^{(n)} < \varepsilon$ $(k \leqslant m_n)$, where ε is a small number fixed in advance. The program composed foresaw the computation of $c_k^{(n)}$ by the given parameters ν and ε through the desirable number of polynomials n, and also the computation of the values of all the polynomials $g_n(\omega)$ at several, arbitrarily chosen points. In the present work we bring up the values of $g_n(\omega)$ at the points $\omega = -1$ and $\omega = 0{,}9$. The first of these points lies at the boundary of the circle of Taylor series convergence of function $g(\omega)$, and the other lies near the singular point $\omega = 1$ of this function.

In the case $\nu = 1$, which corresponds to polynomials of the standard Cauchy-Lipschitz method, the formulas derived are simplified. Namely, at $\nu = 1$

$$d_k^{(m)} = C_m^k, \quad k_1 = \left[\frac{m}{2}\right], \quad k_2 = \min\{m,\ 2^n - 1\}. \tag{31}$$

The computations were conducted at the outset according to a program especially designed for that case. Certain results of these calculations consist in the number k^* of coefficients $c_k^{(n)}$, corresponding to the limit $\varepsilon = 10^{-9}$, and the values of $g_n(\omega)$ for $\omega = -1$ and $\omega = 0.9$ are compiled in Table 1.As in most of the subsequent tables, the following order of number writing is admitted: sign of the number, sign of the order, order, mantissa. As may be seen, the values $g_n(-1)$ and $g_n(0.9)$ converge very slowly to the limit values $g(-1) = 0.5$ and $g(0.9) = 10$. This was precisely the compelling reason for us to abandon the usual Cauchy-Lipschitz method and search for more effective $c_k^{(n)}$ with the aid of the Picon modification.

The coefficients $c_k^{(n)}$ of the first eight polynomials $g_n(\omega)$ for the values of ν from 1 to 9 were computed by the general program. For the values $\nu = 10$ and $\nu = 11$ the coefficients $c_k^{(n)}$ were computed for the first twenty polynomials $g_n(\omega)$. We assumed everywhere $\varepsilon = 10^{-10}$. It may be seen from Table 2 how the number of coefficients $c_k^{(n)}$ increases with the rise of n and ν. In Table 3 we compiled the values of $g_n(-1)$ and $g_n(0.9)$ for $n = 1, 2, \ldots, 8$ and $\nu = 2, 3, \ldots, 9$. The values of $g_n(\omega)$ for $\nu = 1$, computed according to the general program with $\varepsilon = 10^{-10}$, coincided with the corresponding values of Table 1. Finally, given in Table 4 are the values of $g_n(\omega)$ for $\nu = 10$ and $\nu = 11$, and also the number of terms k^* in the corresponding polynomials. It is interesting to note that the polynomials $g_n(-1)$ with odd ν approach $g(-1)$ from below, and $g_n(-1)$ with even ν — from above.

From the analysis of these data it follows that the polynomials $g_n(\omega)$ with great ν and small n are considerably more effective than the polynomials with small ν and great n. At the same time it is not advantageous to take too great ν on account of the large number of terms in the corresponding polynomials. The choice of required polynomials was influenced also by the circumstance that the program for the computation of coefficients a_k in the problem of three bodies allowed us to determine these coefficients for $k = 0, 1, \ldots, 157$. This is why for the subsequent work we selected and recorded on a magnetic tape the coefficients $c_k^{(n)}$ $(n = 2, 3, 4, 5)$ for the values $\nu = 9$ and $\nu = 10$.

TABLE 1

Values and Number of Terms in Polynomials $g_n(\omega)$ for $\nu = 1$ and $\varepsilon = 10^{-9}$

n	k^*	$g_n(-1)$	$g_n(0.9)$	n	k^*	$g_n(-1)$	$g_n(0.9)$
1	2	++00 000000000	++01 190000000	21	89	++00 491539326	++01 582226813
2	4	375000000	239612500	22	93	491933266	591047720
3	8	428898034	278657470	23	97	492292147	599475887
4	16	449836998	311573145	24	101	492620449	607539018
5	22	461170436	340259517	25	104	492921923	615262079
6	27	468306476	365767913	26	108	493199729	622667648
7	32	473219912	388763313	27	111	493456548	629776210
8	37	476811488	409702747	28	115	493694673	636606412
9	41	479552248	428918640	29	119	493916073	643175275
10	45	481712879	446662933	30	122	494122450	649498379
11	50	483460124	463132605	31	126	494315284	655590018
12	54	484902359	478485476	32	129	494495867	661463338
13	58	486113092	492850487	33	133	494665328	667130461
14	62	487143953	506334707	34	136	494824665	672602578
15	66	488032273	519028238	35	140	494974759	677890048
16	70	488805718	531007784	36	143	++00 495116393	++01 683002475
17	74	489485225	542339284				
18	78	400086934	553079912				
19	82	490623486	563279608				
20	86	++00 491104923	++01 572982292				

TABLE 2

Number of Terms in Polynomials $g_n(\omega)$ for $\varepsilon = 10^{-10}$

n/ν	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	4	9	16	25	36	44	51	57	63	66	68
3	8	26	41	55	68	78	86	94	100	105	108
4	16	38	61	80	94	107	116	123	129	134	138
5	23	51	79	103	119	130	140	147	153	158	162
6	29	63	98	124	141	153	162	168	174	179	183
7	34	76	115	144	161	172	181	188	193	198	201
8	39	88	131	161	178	190	198	204	210	214	218

TABLE 3

Values of Polynomials $g_n(\omega)$ for $\nu = 2, 3, \ldots, 9$ and $\varepsilon = 10^{-10}$

$g_n(-1)$

n/ν	2	3	4	5
1	++01 100000000	++00 000000000	++01 100000000	++00 000000000
2	++00 574218750	471649170	++00 514083565	493500958
3	524350346	493579552	501950091	499410014
4	511791984	497654126	500512257	499886133
5	506913968	498902382	500187179	499967246
6	504534706	499402706	500083567	499987961
7	503200602	499640263	500042670	499994779
8	++00 502378790	++00 499767001	++00 500023986	++00 499997450

TABLE (3) (continuation)

$g_n(-1)$

n/ν	6	7	8	9
1	+-+01 100000000	+-+00 000000000	+-+01 100000000	+-+00 000000000
2	+-+00 503166812	498460082	+-+00 500757955	499625320
3	500184332	499941564	500018781	499993906
4	500026047	499993924	500001440	499999654
5	500005896	499998916	500000203	499999961
6	500001784	4999[illegible]729	500000042	499999993
7	500000657	4999[illegible]915	500000012	499999999
8	+-+00 500000279	+-+00 499999969	+-+00 500000004	+-+00 500000000

$g_n(0.9)$

n/ν	2	3	4	5
1	+-+01 271000000	+-+01 343900000	+-+01 409510000	+-+01 468559000
2	379513669	502745097	603548923	683191462
3	465154736	618217848	728951944	805671402
4	534750921	702688863	810302400	876649253
5	592165371	765175647	864131586	919151447
6	640052359	812026493	900580935	945453072
7	680359149	847645055	925825546	962258737
8	+-+01 714559216	+-+01 875091860	+-+01 943681984	+-+01 973280487

$g_n(0.9)$

n/ν	6	7	8	9
1	+-+01 521703100	+-+01 569532790	+-+01 612579511	+-+01 651321560
2	745527124	794509173	833340668	864407012
3	858872447	896390129	923338073	942982300
4	918170518	944881453	962488431	974301658
5	950618935	969268937	980639191	987707576
6	969134655	982151385	989534466	993809779
7	980101908	989255838	994109619	996739593
8	+-+01 986816576	+-+01 993325074	+-+01 996564564	+-+01 998213453

TABLE 4

Values and Number of Terms in Polynomials $g_n(\omega)$ for

$\nu = 10$, $\nu = 11$ $\qquad \varepsilon = 10^{-10}$

ν	n	k^*	$g_n(-1)$	$g_n(0.9)$	ν	n	k^*	$g_n(-1)$	$g_n(0.9)$
	1	11	+-+01 100000000	+-+01 6861894[illegible]		1	12	+-+00 000000000	+-+01 717570464
	2	66	+-+00 500185921	889446847		2	68	499907501	909738646
	3	105	500001992	957452897		3	108	499999346	968186289
	4	134	500000083	982321858		4	138	499999979	987808189
	5	158	500000008	992157639		5	162	499999999	994981748
	6	179	500000001	996318569		6	183	500000000	997803098
	7	198	500000000	998184402		7	201	500000000	998985047
	8	214	500000000	999064753		8	218	500000000	999508233
	9	229	500000000	999499332		9	233	500000000	999751510
	10	243	500000000	999722580		10	247	500000000	999869609
10	11	256	500000000	999841422	11	11	260	500000000	999929186
	12	270	500000000	999906839		12	273	500000000	999960402
	13	282	500000000	999943836		13	285	500000000	999977202
	14	294	500000000	999965370		14	297	500000000	999986558
	15	305	500000000	999978167		15	308	500000000	999991863
	16	316	500000000	999985977		16	319	500000000	999994980
	17	326	500000000	999990809		17	329	500000000	999996818
	18	336	500000[illegible]	999993883		18	339	500000000	999997953
	19	346	500[illegible]	999995870		19	349	500000000	999998664
	20	357	+-+00 500000000	+-+01 999997173		20	359	+-+00 500000000	+-+01 999999117

Section 2. Series of Polynomials in the Two-Body Problem

The problem of two bodies may serve as the simplest example of polynomial series application in celestial mechanics. For the purpose of definiteness let us consider the case of a nondegenerate elliptical motion. The relative coordinates of the bodies represent in themselves analytical functions of time t, or, which in fact is the same, of the mean anomaly M. The disposition of these functions' singularities on the plane M was first studied by Moulton in 1903 [15] who has shown that the coordinates of singular points depend only on eccentricity e and are determined by the equality

$$M = 2k\pi \pm i\Omega \quad (k = 0, \pm 1, \pm 2, \ldots), \tag{32}$$

where

$$\Omega = -\sqrt{1-e^2} + \ln \frac{1+\sqrt{1-e^2}}{e}. \tag{33}$$

Forging somewhat ahead, let us point out that the values of Ω for equidistant values of eccentricity are compiled in Table 5.

Since Ω is rigorously > 0 for any e < 1, the series of polynomials by M for the coordinates of the elliptical problem of two bodies converge for any real M. The constuction of these series will be started with the search for the coefficients of the corresponding Taylor series. Without generality limitation it is sufficient to consider only two functions

$$X = \cos E - e, \quad Y = \sqrt{1-e^2} \sin E, \tag{34}$$

clearly given with the aid of the eccentric anomaly E. Note, by the way, that these functions are tabulated by arguments of M and e (Innes, 1927 [9]). Inasmuch as E is linked with M by the Keplerian equation

$$E - e \sin E = M, \tag{35}$$

the Taylor series of these functions have the form

$$X = \sum_{k=0}^{\infty} a_k M^{2k}, \quad Y = \sum_{k=0}^{\infty} b_k M^{2k+1}. \tag{36}$$

For the determination of a_k and b_k we shall substitute series (36) into the differential equations satisfied by the functions

$$\left.\begin{aligned} (1-e^2-eX)\frac{dX}{dM} &= -\frac{Y}{\sqrt{1-e^2}}, \\ (1-e^2-eX)\frac{dY}{dM} &= \sqrt{1-e^2}\,(e+X). \end{aligned}\right\} \tag{37}$$

Introducing at first the scalar multiplier $\varkappa$ so that

$$a_k = \varkappa^k a_k^*, \quad b_k = \varkappa^k b_k^*, \tag{38}$$

we shall obtain the following system of recurrent formulas:

$$\left.\begin{aligned}(2k+1)(1-e)\,b_k^* &= e\sum_{j=0}^{k-1}(2j+1)\,b_j^* a_{k-j}^* + \sqrt{1-e^2}\,a_k^*,\\ (2k+2)(1-e)\,a_{k+1}^* &= e\sum_{j=0}^{k-1}(2j+2)\,a_{j+1}^* a_{k-j}^* - \frac{b_k^*}{\varkappa\sqrt{1-e^2}}\\ &(k=1,\ 2,\ \ldots),\end{aligned}\right\} \tag{39}$$

allowing us to compute in sequence all a_k^* and b_k^* by the initial coefficients

$$a_0^* = 1-e, \quad b_0^* = \sqrt{\frac{1+e}{1-e}}, \quad a_1^* = -\frac{1}{2(1-e)^2\varkappa}. \tag{40}$$

Having determined the coefficients a_k^* and b_k^* for the given value of $\underline{e}$, we shall find the sequences of polynomials (2)

$$X^{(n)} = \sum_{k=0}^{\left[\frac{m_n}{2}\right]} c_{2k}^{(n)} a_k^* (\sqrt{\varkappa}M)^{2k}, \quad Y^{(n)} = \frac{1}{\sqrt{\varkappa}} \sum_{k=0}^{\left[\frac{m_n-1}{2}\right]} c_{2k+1}^{(n)} b_k^* (\sqrt{\varkappa}M)^{2k+1}. \tag{41}$$

Generally speaking, it is possible to find the letter expressions of a_k and b_k as a function of $\underline{e}$. Indeed, assuming

$$a_k = \frac{(-1)^k \tilde{a}_k}{(2k)!\,(1-e)^{3k-1}}, \quad b_k = \frac{(-1)^k\sqrt{1-e^2}\,\tilde{b}_k}{(2k+1)!\,(1-e)^{3k+1}}, \tag{42}$$

we obtain from (39) at $\varkappa = 1$

$$\left.\begin{aligned}\tilde{a}_k &= e\sum_{j=1}^{k-1} C_{2k-1}^{2j-1}\tilde{a}_j\tilde{a}_{k-j} + \tilde{b}_{k-j},\\ \tilde{b}_k &= e\sum_{j=1}^{k-1} C_{2k}^{2j}\tilde{b}_j\tilde{a}_{k-j} + \tilde{a}_k.\end{aligned}\right\} \tag{43}$$

Hence it may be seen that $\tilde{a}_k$ and $\tilde{b}_k$ $(k = 1, 2, \ldots)$ are polynomials from $\underline{e}$ power $k-1$ with integral positive coefficients

$$\left.\begin{aligned}\tilde{a}_k &= a_0^{(k)} + a_1^{(k)}e + \ldots + a_{k-1}^{(k)}e^{k-1},\\ \tilde{b}_k &= b_0^{(k)} + b_1^{(k)}e + \ldots + b_{k-1}^{(k)}e^{k-1}.\end{aligned}\right\} \tag{44}$$

For the coefficients of these polynomials we may derive the following expressions:

$$\left.\begin{aligned}a_j^{(k+1)} &= \sum_{\lambda=0}^{j}(-1)^{j-\lambda}(\lambda+1)\,2^{-\lambda}C_{3k+2}^{j-\lambda}d_{\lambda,\,k},\\ b_j^{(k)} &= \sum_{\lambda=0}^{j}(-1)^{j-\lambda}\,2^{-\lambda}C_{3k+1}^{j-\lambda}d_{\lambda,\,k},\end{aligned}\right\} \tag{45}$$

where

$$d_{\lambda,k} = \sum_{s=0}^{\left[\frac{\lambda}{2}\right]} \frac{(-1)^s (\lambda+1-2s)^{\lambda+2k+1}}{s!\,(\lambda+1-s)!}. \tag{46}$$

The first few polynomials computed by these formulas will be

$$\left.\begin{array}{ll}
\tilde{a}_1 = 1, & \tilde{b}_1 = 1, \\
\tilde{a}_2 = 1 + 3e, & \tilde{b}_2 = 1 + 9e, \\
\tilde{a}_3 = 1 + 24e + 45e^2, & \tilde{b}_3 = 1 + 54e + 225e^2, \\
\tilde{a}_4 = 1 + 117e + 1107e^2 + 1575e^3, & \tilde{b}_4 = 1 + 243e + 4131e^2 + 11025e^3. \\
\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots
\end{array}\right\} \tag{47}$$

Although formulas (42), (44) - (46) allow us to find the letter values of the coefficients a_k and b_k, their utilization for computer calculations is hardly appropriate. Note that for that purpose the Stumpff formulas [26] are also of little convenience; they have an entirely different structure, but they also allow us to find the general terms of Taylor series of the two-body problem.

According to the program drawn, at first a_k^* and b_k^* were computed for any $\underline{e}$ and κ by formulas (39), then polynomials (41) were computed for the values of M $M = \kappa^{-1/2}$. At the same time we utilized the coefficients $c_k^{(n)}$ (n = 2, 3, 4, 5) for ν = 9 and ν = 10, indicated in the preceding section. Moreover, the exact values of X(M) and Y(M), obtained by way of the solution of the Keplerian equation, were also computed. The results of these calculations are compiled in Table 5.

In Table 5 [following pages] the values of X(M) and Y(M) and the corresponding sequences of polynomials (41) are given for each e = 0.05(0.05)0.95 and the values M = Ω and M = 1.1Ω. For M = Ω, that is, at the boundary of the circle of series'(36) convergence, polynomials with n = 5 give a practically exact result, say a coincidence of eight-nine significant numerals. For M = 1.1Ω such a precision is not attained here, for an insufficient number of terms was retained in polynomials $g_n(\omega)$; on the strength of this polynomials with n = 5 provide a precision by one order lesser than the polynomials of the preceding approximation, namely with n = 4.

In reality, because of the insufficient number of coefficients $c_k^{(n)}$ bounded by the value $\varepsilon = 10^{-10}$, we are compelled to reject for the terms $a_k \omega^k$ rising in absolute value and as n increases, the terms $c_k^{(n)} a_k \omega^k$, so much the greater in absolute value that the number n of the polynomial is greater. On account of that, the polynomial's (41) sequences, compiled in Table 6 for M = 1.1Ω end up with the number n = 4.

As already indicated, it is not difficult to extend the computation of $c_k^{(n)}$ till as small an ε as is desirable. Then the sequences of polynomials (41) may also be applied for great values of M.

TABLE 5

Convergence of Sequences of Polynomials in the Two-Body Problem

e		0.05	0.10	0.15	0.20	0.25
$M = \Omega$		+-+01 268950465	+-+01 199823541	+-+01 159590810	+-+01 131263577	+-+01 109519123
$X\ (M)$		—+00 958468872	—+00 592093808	——00 322916088	——00 141539652	——01 197627633
$X^{(2)}$	$\nu = 9$	958199675	591977084	322946551	141483454	197284100
$X^{(3)}$	$\nu = 9$	958468524	592093646	322915993	141539588	197627169
$X^{(4)}$	$\nu = 9$	958468972	592093851	322916113	141539668	197627759
$X^{(5)}$	$\nu = 9$	958468892	592093817	322916093	141539656	197627659
$X^{(2)}$	$\nu = 10$	958574814	592139531	322943331	141549143	197762224
$X^{(3)}$	$\nu = 10$	958469925	592094262	322916359	141539836	197628972
$X^{(4)}$	$\nu = 10$	958468825	592093831	322916102	141539661	197627700
$X^{(5)}$	$\nu = 10$	958468877	592093811	322916090	141539653	197627640
$Y\ (M)$		+—00 417429752	+-+00 866178531	+-+00 973948782	+-+00 978119666	+-+00 942233458
$Y^{(2)}$	$\nu = 9$	417628485	866264309	973999897	978154360	942258782
$Y^{(3)}$	$\nu = 9$	417433469	866180134	973949738	978120316	942233931
$Y^{(4)}$	$\nu = 9$	417429995	866178636	973948844	978119709	942233488
$Y^{(5)}$	$\nu = 9$	417429720	866178543	973948789	978119672	942233461
$Y^{(2)}$	$\nu = 10$	417584242	866245286	973988532	978146689	942253068
$Y^{(3)}$	$\nu = 10$	417430208	866178727	973948899	978119746	942233515
$Y^{(4)}$	$\nu = 10$	417429744	866178527	973948780	978119665	942233457
$Y^{(5)}$	$\nu = 10$	417429750	866178530	973948782	978119666	942233458
$M = 1.1\ \Omega$		+-+01 295845511	+-+01 219805895	+-+01 175549891	+-+01 144369935	+-+01 120471035
$X\ (M)$		—+01 103482062	—+00 746905651	——00 470981288	——00 272512821	——00 132453258
$X^{(2)}$	$\nu = 9$	103404818	746570277	470781408	272377111	132354457
$X^{(3)}$	$\nu = 9$	103481822	746904564	470980642	272512384	132452940
$X^{(4)}$	$\nu = 9$	103482074	746905699	470981317	272512841	132453273
$X^{(2)}$	$\nu = 10$	103520594	747073926	471080349	272580056	132502193
$X^{(3)}$	$\nu = 10$	103482357	746906919	470982044	272513335	132453632
$X^{(4)}$	$\nu = 10$	103482054	746905618	470981269	272512808	132153249
$Y\ (M)$		+—00 173358054	+-+00 758747618	+-+00 936370207	+-+00 977216565	+-+00 961533328
$Y^{(2)}$	$\nu = 9$	173987066	759018989	936531832	977326226	961613101
$Y^{(3)}$	$\nu = 9$	173367093	758751519	936372532	977218143	961534476
$Y^{(4)}$	$\nu = 9$	173358508	758747815	936370324	977216645	961533386
$Y^{(2)}$	$\nu = 10$	173789176	758934046	936481250	977291902	961588129
$Y^{(3)}$	$\nu = 10$	173359681	758748322	936370627	977216850	961533535
$Y^{(4)}$	$\nu = 10$	173358154	758747661	936370233	977216582	961533340

T A B L E 5 (continued)

e		0.80	0.85	0.90	0.95
$M = \Omega$		+—01 931471803	+—01 588989444	+—01 312554136	+—01 107865391
$X\ (M)$		+—00 114062710	+—01 887559399	+—01 611055710	+—01 314321347
$X^{(2)}$		114045204	887576901	611066672	314326514
$X^{(3)}$	$\nu = 9$	114062713	887559420	611055724	314321354
$X^{(4)}$		114062709	887559391	611055705	314321345
$X^{(5)}$		114062709	887559397	611055709	314321347
$X^{(2)}$		114061732	887552539	611051415	314319323
$X^{(3)}$	$\nu = 10$	114062700	887559328	611055667	314321327
$X^{(4)}$		114062709	887559394	611055708	314321346
$X^{(5)}$		114062710	887559398	611055710	314321347
$Y\ (M)$		+—00 243343728	+—00 181520237	+—00 120384614	+—01 598925627
$Y^{(2)}$		243345549	181521514	120385413	598929391
$Y^{(3)}$	$\nu = 9$	243343763	181520261	120384629	598925698
$Y^{(4)}$		243343731	181520239	120384615	598925632
$Y^{(5)}$		243343729	181520238	120384614	598925627
$Y^{(2)}$		243345140	181521228	120385234	598928546
$Y^{(3)}$	$\nu = 10$	243343733	181520240	120384616	598925636
$Y^{(4)}$		243343728	181520237	120384614	598925627
$Y^{(5)}$		243343728	181520237	120384614	598925627
$M = 1.1\ \Omega$		+—00 102461898	+—01 647888389	+—01 343809550	+—01 118651930
$X\ (M)$		+—01 997711056	+—01 785391671	+—01 545993255	+—01 283185056
$X^{(2)}$		997782900	785442091	546024831	283199937
$X^{(3)}$	$\nu = 9$	997711284	785391829	545993354	283185100
$X^{(4)}$		997711044	785391662	545993248	283185050
$X^{(2)}$		997675536	785366744	545977644	283177696
$X^{(3)}$	$\nu = 10$	997710783	785391478	545993133	283184996
$X^{(4)}$		997711062	785391674	545993256	283185054
$Y\ (M)$		+—00 261817311	+—00 195559880	+—00 129848466	+—01 646688325
$Y^{(2)}$		261823059	195563911	129850989	646700213
$Y^{(3)}$	$\nu = 9$	261817394	195559938	129848503	646688501
$Y^{(4)}$		261817316	195559883	129848468	646688339
$Y^{(2)}$		261821259	195562648	129850199	646696192
$Y^{(3)}$	$\nu = 10$	261817326	195559890	129848473	646688361
$Y^{(4)}$		261817312	195559880	129848467	646688332

Section 3. The Sundman Series in the Two-Body Problem

The Sundman series coefficients in the two-body problem may be obtained in a final form. Indeed, assume that function $f(\omega)$, given in the form (1), is analytical in an infinite band 2Ω wide and symmetrical relative to the real axis ω. After applying the Poincaré transformation (9) this function may be expanded in power series

$$f(\theta) = a_0 + \sum_{s=1}^{\infty} A_s \theta^s, \tag{48}$$

converging in a circle $|\theta| < 1$. As was shown in the above-mentioned work by Belorizky, in order to find the coefficients A_S, it is sufficient to substitute in (1) the expansion of ω by powers θ, stemming from (9). We then obtain

$$A_s = \sum_{k=0}^{\left[\frac{s-1}{2}\right]} a_{s-2k} p^{(s)}_{s-2k} \left(\frac{4\Omega}{\pi}\right)^{s-2k}. \tag{49}$$

Here $p^{(s)}_{s-2k}$ are positive numbers, which are coefficients of the expansion

$$\left(\theta + \frac{\theta^3}{3} + \frac{\theta^5}{5} + \dots\right)^q = \sum_{s=0}^{\infty} p_q^{(q+2s)} \theta^{q+2s}. \tag{50}$$

Belorizky indicated also the recurrent formulas

$$sp^{(s)}_{s-2k} = (s-2k) p^{(s-1)}_{s-1-2k} + (s-2) p^{(s-2)}_{s-2k}$$
$$\left(s = 3, 4, \dots;\ k = 0, 1, \dots, \left[\frac{s-1}{2}\right]\right), \tag{51}$$

allowing to compute these coefficients in sequence by the initial values $p_1^{(1)} = p_2^{(2)} = 1$.

Therefore, if the coefficients of the Taylor series (1) of function $f(\omega)$ are known in a final form, the Sundman series coefficients (48) may also be found by formula (49) in the final form. This is why the knowledge of the common terms a_k and b_k of series (36) allows us to write also the common terms A_k and B_k of the Sundman series of two bodies

$$X = \sum_{k=0}^{\infty} A_k \theta^{2k}, \quad Y = \sum_{k=0}^{\infty} B_k \theta^{2k+1}. \tag{52}$$

We have computed all the coefficients $p^{(s)}_{s-2k}$ through the number $s = 120$ inclusive. However, they were without use, for it was found to be simpler to find A_k and B_k directly, without utilizing their relationship with a_k and b_k.

Indeed, as functions of θ, X and Y satisfy the equations

$$\left.\begin{aligned}(1-e^2-eX)\frac{dX}{d\theta}&=-\frac{4\Omega}{\pi\sqrt{1-e^2}}\,\frac{Y}{1-\theta^2},\\(1-e^2-eX)\frac{dY}{d\theta}&=\frac{4\Omega\sqrt{1-e^2}}{\pi}\,\frac{e+X}{1-\theta^2}.\end{aligned}\right\}\qquad(53)$$

Hence follow the recurrent formulas for the coefficients A_k and B_k

$$\left.\begin{aligned}(2k+1)(1-e)\,B_k&=e\sum_{j=0}^{k-1}(2j+1)\,B_jA_{k-j}+\frac{4\Omega\sqrt{1-e^2}}{\pi}\left(1+\sum_{j=0}^{k-1}A_{j+1}\right),\\(2k+2)(1-e)\,A_{k+1}&=e\sum_{j=0}^{k-1}(2j+2)\,A_{j+1}A_{k-j}-\frac{4\Omega}{\pi\sqrt{1-e^2}}\sum_{j=0}^{k}B_j\end{aligned}\right\}\qquad(54)$$

with the initial conditions

$$A_0=1-e,\quad B_0=\frac{4\Omega}{\pi}\sqrt{\frac{1+e}{1-e}},\quad A_1=-\frac{8\Omega^2}{\pi^2(1-e)^2}.\qquad(55)$$

The following actions were taken according to the program prepared:

1) the calculation of A_{k+1} and B_k by formulas (54) to k = 1700 inclusive;
2) summation of series (52) for the values θ = 0.05 (0.05) 0.95, whereupon this summation continued till the simultaneous fulfillment of the conditions $|A_k\theta^{2k}|<10^{-10}$, $|B_k\theta^{2k+1}|<10^{-10}$;
3) calculation of M and of the exact values of X(M), Y(M) for these values
4) summation of series (52) for the value of θ corresponding to M = 2π, till the fulfillment of the above-indicated conditions, or to k = 1700, if these conditions are not satisfied.

All these operations were performed for every e = 0.05 (0.05) 0.95. Part of the results obtained is reflected in the tables presented here. The value of the mean anomaly M is given in Table 6 as a function of e and θ. It is important to note that to the value M = Ω, that is, to the radius of series' (36) convergence, corresponds for any e one and the same value

$$\theta=\frac{\exp(\pi/2)-1}{\exp(\pi/2)+1}\approx 0.65579\,42026.\qquad(56)$$

The values of X(θ), Y(θ) of series (52) for all θ from 0.05 to 0.95 coincided with the exact values of X(M), Y(M). The number k* of the last retained term in this series is brought out in Table 7, from which it is clear that the Sundman series coefficients vary very little as a function of eccentricity, inasmuch as the number of terms in these series is mainly determined by the value of θ only. Compiled in Table 8 are the values of X(θ) and Y(θ) of series (52) for M = 2 , the corresponding values of θ and the number k*. The exact values of X(M), Y(M) in this case will be 1 - e and 0. It may be seen that for e = 0.15, the 1700 terms of Sundman series were already found to be insufficient to assure the precision in nine decimal signs and for e = 25, even the first ones were already wrong.

Section 4. Series of Polynomials in the Problem of Three Bodies

The determination of coefficients a_k in the problem of three bodies in letter form and, by the same token, the analytical determination of the corresponding coefficients of Sundman series, would have a very great significance and, in particular, as was shown by Belorizky in [5], to allow basically the solution of the question of stability by Lagrange. Unfortunately, even the numerical determination of coefficients a_k for concrete initial conditions is linked with fairly considerable difficulties. Indeed, the finding of these coefficients by way of consecutive differentiation of the right-hand parts of equations of motion is in practice totally unfeasible. The way out of this situation was shown by Steffensen [24], who proposed to reduce the equations to second power independently from their order, and then obtain recurrent relations for a_k. Steffensen utilized the power series by t for the representation of the solution of the three-body problem in a certain neighborhood of the initial moment. Later, Rauch [22] and Rauch and Riddel [23] applied the Steffensen method to the problem of n bodies, whereupon in the first of these works the time t for taken for the independent variable, and in the second — the regularizing variable ω.

We shall consider the equations of the problem of three bodies m_1, m_2, m_3 in relative coordinates $r_1 = \overrightarrow{m_2 m_3}$, $r_2 = \overrightarrow{m_3 m_1}$, $r_3 = \overrightarrow{m_1 m_2}$

$$\ddot{r}_i = -fM\frac{r_i}{r_i^3} + fm_i\left(\frac{r_1}{r_1^3} + \frac{r_2}{r_2^3} + \frac{r_3}{r_3^3}\right) \quad (i = 1, 2, 3). \tag{57}$$

Here M is the sum of masses, f is the gravitational constant and

$$r_1 + r_2 + r_3 = 0. \tag{58}$$

With aid of the force function

$$U = f\left(\frac{m_2 m_3}{r_1} + \frac{m_3 m_1}{r_2} + \frac{m_1 m_2}{r_3}\right) \tag{59}$$

and of the scalar multiplier κ, we shall introduce the regularizing variable by the formula

$$d\omega = \kappa U dt \tag{60}$$

whereupon we shall consider that at $t = 0$, $\omega = 0$. Denoting by a prime the differentiation with respect to ω, we shall reduce Eq.(57) to the form

$$r_i'' + \frac{U'}{U} r_i' = \frac{1}{\kappa^2 U^2}\left[-fM\frac{r_i}{r_i^3} + fm_i\left(\frac{r_1}{r_1^3} + \frac{r_2}{r_2^3} + \frac{r_3}{r_3^3}\right)\right] \quad (i = 1, 2, 3). \tag{61}$$

Let us assume that at the initial moment the following two conditions are fulfilled :

../..

1) $|\bar{C}| > 0$, where

$$\frac{m_2 m_3}{M}[r_1 \times \dot{r}_1] + \frac{m_3 m_1}{M}[r_2 \times \dot{r}_2] + \frac{m_1 m_2}{M}[r_3 \times \dot{r}_3] = C; \tag{62}$$

2) $\min\{r_1, r_2, r_3\} > 0$.

The first condition is sufficient for the elimination of the possibility of triple collisions in the course of the entire time of motion. The second condition, implying the absence of double collision at the initial moment of time, is not compelling and is utilized only for the sake of simplicity. Without this condition the system (61) ought to be reduced to clearly regularized form by introduction of new variables.

Denoting by Δ_i the squares of mutual distances and by σ_i the cubes of the reciprocal values of these distances, we finally obtain the following system of eighteen equations of second order:

$$\left.\begin{array}{ll}
\Delta_i = r_i^2 & (i=1,\ 2,\ 3), \\
2\Delta_i\sigma_i' + 3\sigma_i\Delta_i' = 0 & (i=1,\ 2,\ 3), \\
V r_i'' + \frac{1}{2} V' r_i' = -\frac{fM}{\varkappa^2}\sigma_i r_i + \frac{f m_i}{\varkappa^2}(\sigma_1 r_1 + \sigma_2 r_2 + \sigma_3 r_3) & (i=1,\ 2,\ 3), \\
U = f(m_2 m_3 \sigma_1 \Delta_1 + m_3 m_1 \sigma_2 \Delta_2 + m_1 m_2 \sigma_3 \Delta_3), & \\
V = U^2, & \\
\varkappa U t' = 1 &
\end{array}\right\} \tag{63}$$

for the determination of eighteen unknown functions

$$\left.\begin{array}{llll}
r_i = \sum\limits_{k=0}^{\infty} r_i^{(k)}\omega^k, & \Delta_i = \sum\limits_{k=0}^{\infty} \Delta_i^{(k)}\omega^k, & \sigma_i = \sum\limits_{k=0}^{\infty} \sigma_i^{(k)}\omega^k & (i=1,\ 2,\ 3), \\
U = \sum\limits_{k=0}^{\infty} u^{(k)}\omega^k, & V = \sum\limits_{k=0}^{\infty} v^{(k)}\omega^k, & t = \sum\limits_{k=1}^{\infty} t^{(k)}\omega^k. &
\end{array}\right\} \tag{64}$$

Note that the utilization of relation (58) allows us to reduce the number of unknown functions to fifteen, but it would be more appropriate to keep this relation for the control, and to consider the coefficients $r_1^{(k)}, r_2^{(k)}, r_3^{(k)}$ in the process of computations as independent. According to quantities $r_i(0), \dot{r}_i(0)$ given at the initial moment of time t = 0, we shall find the first coefficients of series (64)

$$\left.\begin{array}{l}
r_i^{(0)} = r_i(0),\ \ \Delta_i^{(0)} = |r_i(0)|^2,\ \ \sigma_i^{(0)} = |r_i(0)|^{-3}, \\
u^{(0)} = f(m_2 m_3 \sigma_1^{(0)}\Delta_1^{(0)} + m_3 m_1 \sigma_2^{(0)}\Delta_2^{(0)} + m_1 m_2 \sigma_3^{(0)}\Delta_3^{(0)}), \\
v^{(0)} = [u^{(0)}]^2,\ \ r_i^{(1)} = \frac{\dot{r}_i(0)}{\varkappa u^{(0)}},\ \ t^{(1)} = \frac{1}{\varkappa u^{(0)}}.
\end{array}\right\} \tag{65}$$

The substitution of (64) into (63) leads to the recurrent formulas for the determination of subsequent coefficients

$$\begin{aligned}
&\Delta_i^{(k)} = \sum_{j=0}^{k} r_i^{(j)} r_i^{(k-j)},\\
&\sigma_i^{(k)} = -\frac{1}{2k\Delta_i^{(0)}} \sum_{j=0}^{k-1} (3k-j)\, \sigma_i^{(j)} \Delta_i^{(k-j)},\\
&u^{(k)} = f \sum_{j=0}^{k} (m_2 m_3 \sigma_1^{(j)} \Delta_1^{(k-j)} + m_3 m_1 \sigma_2^{(j)} \Delta_2^{(k-j)} + m_1 m_2 \sigma_3^{(j)} \Delta_3^{(k-j)}),\\
&v^{(k)} = \sum_{j=0}^{k} u^{(j)} u^{(k-j)}, \qquad\qquad (66)\\
&r_i^{(k+1)} = \frac{1}{k(k+1)v^{(0)}} \sum_{j=0}^{k-1} \Big[-\frac{1}{2}(j+1)(k+j)\, v^{(k-j)} r_i^{(j+1)} -\\
&\quad - \frac{fM}{\varkappa^2} \sigma_i^{(j)} r_i^{(k-j-1)} + \frac{fm_i}{\varkappa^2} (\sigma_1^{(j)} r_1^{(k-j-1)} + \sigma_2^{(j)} r_2^{(k-j-1)} + \sigma_3^{(j)} r_3^{(k-j-1)}) \Big],\\
&t^{(k+1)} = -\frac{1}{(k+1)u^{(0)}} \sum_{j=0}^{k-1} (j+1)\, t^{(j+1)} u^{(k-j)}\\
&\qquad\qquad (k = 1, 2, \ldots).
\end{aligned}$$

According to the established program all the coefficients of series (64) were computed to the number k = 157 inclusively, and then sequences of polynomials (2) were constructed with the aid of the convergence multipliers $c_k^{(n)}$ of the first section (n = 2, 3, 4, 5; ν = 9 and ν = 10).

The coefficients a_k in the three-body problem, that is, the coefficients of series (64) were obtained by us for four examples. These coefficients are denoted in the following respectively as $a_k(1)$, $a_k(2)$, $a_k(3)$ and $a_k(4)$. As a first example we took the Lagrange solutions, but the corresponding coefficients $a_k(1)$ were utilized only for various control actions. As a second example we considered the plane hyperbolic-elliptic-type motion studied by Zumkley in 1941 [31] by the numerical integration method. In this motion all the three masses are postulated equal to unity, and the initial conditions are:

$$\left.\begin{array}{ll}
\vec{r}_1 = (\ \ 2.5,\ 0,\ 0), & \dot{\vec{r}}_1 = (0,\ \ \ 2.5,\ 0),\\
\vec{r}_2 = (-1.5,\ 0,\ 0), & \dot{\vec{r}}_2 = (0,\ -1,\ \ \ 0),\\
\vec{r}_3 = (-1,\ \ \ 0,\ 0), & \dot{\vec{r}}_3 = (0,\ -1.5,\ 0).
\end{array}\right\} \qquad (67)$$

In the Zumkley work the values of $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are given with three marks after coma for t = 0 (02) 2.8 and t = 2.8 (0.1) 10. During that time the mass m_2 effects about 2.5 convolutions around the mass m_1, and the mass m_3 drifts away from the first two along a hyperbolic-type curve.

The third example is based upon the Strömgen work of 1909 [21]. In this case $m_1 = m_2 = 1$, $m_3 = 2$ and at the moment of time t = 0

../..

$$\left.\begin{array}{ll} \bar{r}_1=(-10,\ 0,\ 0), & \dot{\bar{r}}_1=\left(0,\ -\sqrt{\frac{3}{10}},\ 0\right), \\ \bar{r}_2=(-\ 7,\ 0,\ 0), & \dot{\bar{r}}_2=\left(0,\ -\sqrt{\frac{3}{7}},\ 0\right), \\ \bar{r}_3=(\ 17,\ 0,\ 0), & \dot{\bar{r}}_3=\left(0,\ \sqrt{\frac{3}{10}}+\sqrt{\frac{3}{7}},\ 0\right). \end{array}\right\} \tag{68}$$

The following denotations were adopted in the Strömgren work: $m_1=A$, $m_2=B$, $m_3=C$, $\bar{r}_1=-\bar{r}_B$, $\bar{r}_2=\bar{r}_A$, and t = −0.5 was taken for the zero moment of time; the values of $\bar{r}_A$ and $\bar{r}_B$ for t = 0.5 (1) 215.5 were given with a precision 6 to three decimals after coma. During that moment of time the mass m_2 performs four convolutions around the mass m_1, and the mass m_3 drifts away from them along a strongly elongated elliptical-type curve.

Finally, as a fourth example a plane motion with close double rapprochements was taken, which was investigated by Burrau in 1913 [6]. Here $m_1 = 5$, $m_2 = 4$, $m_3 = 3$ and for t = 0

$$\left.\begin{array}{ll} \bar{r}_1=(\ 3,\ -4,\ 0), & \dot{\bar{r}}_1=(0,\ 0,\ 0), \\ \bar{r}_2=(\ 0,\ \ \ 4,\ 0), & \dot{\bar{r}}_2=(0,\ 0,\ 0), \\ \bar{r}_3=(-3,\ \ \ 0,\ 0), & \dot{\bar{r}}_3=(0,\ 0,\ 0). \end{array}\right\} \tag{69}$$

Inasmuch as in this motion the constant $\bar{C}$ of the area integral is zero, and by the same token the possibility is not excluded of a triple collision, one may not assert that the polynomial series converge in this case for any real t. But it was interesting to apply the polynomial series to this type of motion also. Burrau provides the values of $\bar{r}_2$ and $\bar{r}_3$ with a precision to 4-5 decimals after coma for t, varying irregularly from 0 to 3.35. At t = 1.88, there takes place a close rapprochement of m_1 and m_2, and for t = 2.9 — a close rapprochement of m_1 and m_3.

The sequences of polynomials were constructed for all three indicated cases for various values of κ. The values of κ themselves were so assorted that the corresponding coefficients a_k vary sufficiently slowly and that all possible mutual products, figuring in (66), do not come out of the range of numbers represented in the computer M-20. Found subsequently were the values of the constructed polynomials in the series of points ω. The most characteristic results are compiled in Tables 9 - 11.* The data of these tables illustrate the convergence of the sequences of polynomials, interpolated after the result of Zumkley, Strömgren and Burrau at corresponding moments of time. The last numerals of these values are obviously approximate. The value $\omega = 1$ corresponds approximately to $^1/_3$ convolution of m_2 relative to m_1 in the example 2, and $^1/_2$ convolution of m_1 relative to m_3 in the example 3. As may be seen from the tables, the rapidity of convergence of polynomial sequences leaves in these cases nothing to be desired. However, the increase of ω or, which is the same, the decrease of κ at constant $\omega = 1$ leads to a rapid deterioration of convergence. In order to broaden the region of effective application of sequences of polynomials it is necessary, on the one hand, to increase the number of coefficients a_k, and on the other hand, to lower the limit ε, set at calculation of $c_k^{(n)}$

* The values interpolated according to the results of Zumkley, Strömgren and Burrau to the corresponding moments of time are indicated in Tables 9-11 by a star

TABLE 6

Values of the mean anomaly M corresponding to the Poincare Transformation in the two-body problem

e/θ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	0.171	0.344	0.518	0.694	0.875	1.050	1.251	1.451	1.650	1.881	2.118	2.374	2.655	2.970	3.332	3.762	4.302	5.041	6.273
10	127	255	385	516	650	0.787	0.930	1.078	1.233	1.398	1.573	1.761	1.973	2.207	2.475	2.795	3.196	3.746	4.660
15	102	204	307	412	519	629	743	0.861	0.985	1.116	1.257	1.408	1.575	1.762	1.977	2.232	2.552	2.992	3.722
20	084	168	253	339	427	517	611	708	810	0.918	1.034	1.158	1.296	1.450	1.626	1.836	2.099	2.461	3.061
25	070	140	211	283	356	432	510	591	675	766	0.862	0.967	1.081	1.209	1.357	1.532	1.752	2.053	2.554
30	059	118	177	237	299	363	428	496	558	643	724	812	0.908	1.016	1.140	1.287	1.471	1.724	2.145
35	049	099	149	200	252	305	360	416	478	541	609	683	764	0.855	0.959	1.083	1.238	1.451	1.805
40	041	083	125	168	211	256	303	351	401	455	512	574	642	718	805	0.910	1.040	1.219	1.517
45	035	069	105	140	177	214	253	293	336	380	428	480	537	600	673	760	0.870	1.019	1.268
50	029	058	087	116	147	178	210	243	278	315	355	398	445	498	559	631	721	0.845	1.052
55	024	047	071	095	120	146	172	199	228	259	291	326	365	408	458	517	591	693	0.862
60	019	038	057	077	097	118	139	161	184	209	235	254	295	330	370	418	478	560	696
65	015	030	045	061	077	093	110	127	146	165	186	209	233	261	293	330	378	443	551
70	012	023	035	047	059	072	084	098	112	127	143	160	179	200	225	254	290	340	423
75	009	017	026	035	044	053	062	072	083	094	105	118	132	148	166	187	214	251	312
80	006	012	018	024	030	037	043	050	057	065	073	082	092	103	115	130	149	175	217
85	004	008	011	015	019	023	027	032	036	041	046	052	058	065	073	082	094	110	137
90	002	004	005	008	010	012	015	017	019	022	025	028	031	035	039	044	050	059	073
95	001	001	002	003	004	004	005	006	007	008	008	010	011	012	013	015	017	020	025

TABLE 7

Number of terms in the Sundman series of the two-body problem

e/θ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.05	8	10	12	14	16	19	22	25	28	32	37	43	51	61	75	91	131	199	401
10	8	10	12	14	16	18	21	24	27	31	36	42	49	60	74	96	129	196	396
15	7	10	12	14	16	18	21	23	27	31	35	41	49	59	72	93	127	194	396
20	8	9	11	13	15	18	20	23	27	31	35	41	48	58	71	91	125	190	387
25	8	10	11	13	15	17	20	23	26	30	35	40	47	57	71	91	123	189	381
30	8	10	12	13	15	18	20	22	26	30	34	40	47	56	70	89	122	185	375
35	8	10	12	13	15	18	20	22	25	30	34	40	46	56	69	88	120	184	370
40	8	10	12	13	15	18	20	22	26	30	34	40	46	56	68	88	120	184	368
45	8	10	11	13	15	18	20	22	26	30	34	40	46	56	68	88	118	182	366
50	8	10	11	13	15	18	20	22	26	30	34	40	46	56	68	88	118	180	362
55	8	10	11	13	15	18	20	22	26	30	34	40	46	56	68	88	118	180	362
60	7	10	11	13	15	17	20	22	26	29	34	39	46	56	68	88	118	180	362
65	7	9	11	13	15	17	20	22	26	29	34	39	46	56	68	87	118	180	362
70	7	9	11	13	15	17	20	22	26	29	34	39	46	55	68	87	118	180	360
75	7	9	11	13	15	17	20	22	25	29	34	39	46	55	68	86	118	178	360
80	7	9	11	13	15	17	19	22	25	29	33	39	46	54	67	86	117	178	358
85	7	9	11	13	15	17	19	22	25	29	33	38	45	54	67	85	116	176	354
90	7	9	11	13	15	17	19	22	25	28	33	38	45	54	66	84	114	174	350
95	7	9	11	12	14	16	19	21	24	28	32	37	44	53	65	83	112	171	344

TABLE 8

Sundman series of the two body problem for M = 2

e	θ	X (θ)	Y (θ)	k*
0.05	-+-+00 950287098	-+-+00 950000000	+--09 669605829	402
10	-+-+00 985740598	-+-+00 900000000	---08 174681954	1359
15	-+-+00 995845471	-+-+00 850000024	+--07 238028237	1700
20	-+-+00 998905153	-+-+00.808408547	+--01 102373872	1700
25	-+-+00 999756096	-+-+00 941355536	---00 450587949	1700

TABLE 9

Convergence of sequences of polynomials for $\omega = 1$ in the case a_k (2), $1/\kappa = 3.25$

n	t	x_1	y_1	x_2	y_2	x_3	y_3
2	1.810002	0.616739	3.306870	−1.123508	−2.112387	0.506769	−1.194483
3 $\nu = 9$	1.810016	0.616707	3.306863	−1.123587	−2.112464	0.506880	−1.194399
4	1.810016	0.616707	3.306862	−1.123587	−2.112465	0.506880	−1.194397
2	1.810008	0.616695	3.306867	−1.123628	−2.112416	0.506933	−1.194451
3 $\nu = 10$	1.810017	0.616707	3.306862	−1.123588	−2.112465	0.506881	−1.194397
4	1.810016	0.616707	3.306862	−1.123587	−2.112465	0.506880	−1.194397
	1.810016	0.616	3.305	−1.124	−2.114	0.507	−1.191

TABLE 10

Convergence of sequences of polynomials for $\omega = 1$ in the case a_k (3), $1/\kappa = 16.25$

n	t	x_1	y_1	x_2	y_2	x_3	y_3
2	33.289911	−2.769217	−18.344155	6.171048	−2.623783	−3.401831	20.967938
3 $\nu = 9$	33.290375	−2.769527	−18.345743	6.171168	−2.619614	−3.401641	20.965357
4	33.290350	−2.769503	−18.345775	6.171044	−2 619510	−3.401541	20.965284
5	33.290346	−2.769500	−18.345775	6.171030	−2.619506	−3.401529	20.965281
2	33.290493	−2.769972	−18.345172	6.172521	−2.620804	−3.402549	20.965976
3 $\nu = 10$	33.290362	−2.769509	−18.345780	6.171074	−2.619498	−3.401565	20.965278
4	33.290346	−2.769500	−18.345776	6.171030	−2.619504	−3.401530	20.965280
5	33.290345	−2.769500	−18.345775	6.171028	−2.619505	−3.401528	20.965281
	33.290346	−2.769497	−18.345773	6.171027	−2.619508		

TABLE 11

Convergence of sequences of polynomials for $\omega = 1$ in the case a_k (4), $1/\kappa = 40$

n	t	x_1	y_1	x_2	y_2	x_3	y_3
2	1.8587	1.1933	−2.8221	−0.9459	2.8775	−0.2474	−0.0554
3 $\nu = 9$	1.9001	1.0567	−2.7831	−1.0335	2.8127	−0.0232	−0.0296
4	1.8815	1.0548	−2.8114	−1.0519	2.8480	−0.0029	−0.0366
5	1.8799	1.0649	−2.8114	−1.0438	2.8495	−0.0211	−0.0381
•	1.8799			−1.05367	2.84051	+0.00137	−0.01863
2	1.8868	1.1686	−2.7810	−0.9390	2.8242	−0.2296	−0.0432
3 $\nu = 10$	1.8936	1.0402	−2.7966	−1.0551	2.8274	+0.0149	−0.0307
4	1.8784	1.0615	−2.8144	−1.0483	2.8527	−0.0132	−0.0384
5	1.8804	1.0667	−2.8102	−1.0417	2.8483	−0.0250	−0.0381
	1.8804			−1.04898	2.84388	−0.00875	−0.02783

The fourth example was found to be much more complex, as should have been expected. According to data of Table 11, the convergence of the sequences of polynomials is in this case very slow, so that even the second decimal point in polynomials with n = 5 may be erroneous by 1 - 2 units. Such a poor convergence is explained by the following causes. First, the values $\omega = 1$, $1/\kappa = 40$ correspond precisely to the moment of time of close rapprochement of m_1 and m_2, when all the coordinates vary very rapidly. Secondly, by virtue of the irregular character of motion, the coefficients a_k (4) do not vary monotonically as a_k (2) and a_k (3) do. Because of close rapprochement near the initial moment of time the coefficients $u^{(k)}$, $v^{(k)}$ and $\sigma^{(k)}$ for the corresponding number $\underline{i}$ are great in absolute value, and this is why the coefficients $\bar{r}_1^{(k)}$ are computed with a great loss of precision. Undoubtedly, in such cases it is better to bring the system (61) to a clearly regularized form beforehand.

CONCLUSION

The results of this work shows that the series of polynomials may apparently be utilized for the numerical solution of the problem of three bodies. Contrary to the standard numerical integration by steps, the solution is here made in the form of finite analytical expression (2), valid for all ω from zero up to a certain maximum value. This maximum value may be made as great as may be desired by increasing the number $\underline{n}$ of the polynomial and of its power m_n. Obviously, too much may not be expected of polynomials (2), inasmuch as even fast converging power series of trigonometrical functions are effective only at a sufficient proximity to the initial point.

The effectiveness in the utilization of sequences of polynomials representing the general solution of the three-body problem may be improved in numerous ways. First of all, as already pointed out more than once, the number of terms in the polynomials (2) may be significantly increased. After obtaining polynomials (2) they may be subjected to convolution with the aid of Chebyshev polynomials, decreasing in this way their power. Secondly, one may attempt to extend the search for the most effective convergence factors $c_k^{(n)}$. Thirdly, other types of expansions may be tested, for example in series of polynomials in the Mittag-Leffler rectilinear star. Let us recall in this connection the Markushevich expansion of 1944 [1]

$$f_n(\omega) = \sum_{k=0}^{l_n} \gamma_k^{(n)} a_k \omega^k + \sum_{k=0}^{m_n} (1 - \gamma_k^{(n)}) a_k \omega^k, \tag{70}$$

generalizing (2). Here $\gamma_k^{(n)}$ are certain real numbers, such that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|\gamma_k^{(n)}|} = \overline{\lim_{k\to\infty}} \sqrt[k]{|1-\gamma_k^{(n)}|} = 1, \tag{71}$$

and $\{l_n\}$ and $\{m_n\}$ are certain sequences of natural numbers approaching the infinity alongside with $\underline{n}$.

Finally, one more interesting possibility should be mentioned, namely, the representation of the general solution of the three-body problem in the form of series by Hermite polynomials. The convergence region of these series is the band $|\mathrm{Im}(\omega)| <$ const. Without investigating the question of convergence of coordinate expansion in Hermite polynomial series in the three-body problem over the entire analyticity $|\mathrm{Im}(\omega)| < \Omega$, let us only point out that these coordinates satisfy the well known simple and sufficient conditions for the convergence of the series of Hermite polynomials of function $f(\omega)$ over the entire real axis:

1) $F(\omega)$ is a piecewise-smooth function in any finite interval of that axis;

2) the integral $\int_{-\infty}^{\infty} |\omega| f^2(\omega) \exp(-\omega^2)\, d\omega$ has a finite value.

In the expansion by Hermite polynomials, just as also in the expansion in series of polynomials in the rectilinear Mittag-Leffler star, the quantity does not appear anywhere in explicit form, as this takes place in the Sundman series, and this is why one may hope for a more rapid convergence of these expansions by comparison with the convergence of the Sundman series.

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